

Significant Figures and an Introduction to the Normal Distribution

Object: To become familiar with the proper use of significant figures and to become acquainted with some rudiments of the theory of measurement.

Apparatus: Stopwatch, pendulum.

References: Mathematical Preparation for General Physics, Chapter 2, by Marion & Davidson; Introduction to the Theory of Error by Yardley & Beers.

Significant Figures

Laboratory work involves the recording of various kinds of measurements and combining them to obtain quantities that may be compared with a theoretical result. For such a comparison to be meaningful, the experimenter must have some idea of how accurate his measurements are, and should report this along with his data and conclusion.

One obvious limit to attainable accuracy in any instance is the size of the smallest division of the measuring instrument. For example, with an ordinary meter stick, one can measure to within 1 mm. Thus, the result of such a measurement would be given as, e.g., 0.675 m. This result contains three significant figures. It does not convey the same meaning as 0.6750 m which implies that a device capable of measuring to 0.0001 m or 0.1 mm was used, and that the result was as given. If the object were less than 0.1 m long, a measurement might give 0.075 m which contains only 2 significant figures because zeros used to locate the decimal point are not significant figures. For a still smaller object, a one significant figure result might be obtained, e.g. 0.006 m. It is important, in any kind of measurement, to judge all the factors affecting the accuracy of that measurement and to record the data using the appropriate number of significant figures.

When combining quantities, it is possible, using addition and subtraction, to get a result with more or less significant figures than the original quantities had. Thus,

$$\begin{array}{r} 0.721 \text{ m} \\ + \underline{0.675 \text{ m}} \\ 1.396 \text{ m} \end{array}$$

this leads to a 4 significant figure result. On the other hand

$$\begin{array}{r} 0.721 \text{ m} \\ - \underline{0.675 \text{ m}} \\ 0.046 \text{ m} \end{array}$$

and one significant figure has been lost. To add or subtract two quantities of different accuracy, the most accurate must be rounded off. For example, to add 12.3 and 1.57 m, one must first round off 1.57 m to 1.6 m and then add.

To multiply and divide, the rule is that the result must contain the same number of significant figures as that of the original quantity which has the least number. For example, to compute the volume of a piece of sheet metal with dimensions 25.7 cm, 32.6 cm, 0.1 cm, the result is

$$25.7 \text{ cm} \times 32.6 \text{ cm} \times 0.1 \text{ cm} = 8 \times 10^1 \text{ cm}^3$$

The exponential form of expressing the result must be used because to write 80 cm^3 would imply 2 significant figures.

Procedure:

Do exercises as provided by the laboratory instructor.

Propagation of Uncertainties:

Numerical data developed during the course of an experiment should be accompanied by an estimate of the uncertainty or error. In the simplest case where only one measurement is involved, the uncertainty should be taken as the smallest unit of the measuring device. For example, the result of a length measurement with a meter stick might be

$$(0.323 \pm 0.001)m$$

If repeated measurements of the same quantity are taken, the proper uncertainty to use is the “standard deviation of the mean” which is discussed in the next section. It is frequently necessary to combine numbers with their uncertainties to arrive at a final result. The procedures for doing this will now be presented.

Suppose that in measuring the duration of a phenomenon, one can come no closer than ± 4 s, for whatever reason. The result might be stated (18 ± 4) s. The question arises: What is the proper limit of uncertainty to put on the sums or difference of two such measurements? Thus

$$(18 \pm 4) \text{ s} + (23 \pm 3) \text{ s} = ?$$

It would not seem reasonable to either add the uncertainties, getting ± 7 s, or to subtract them, getting ± 1 s, so the proper result must be somewhere in between. From a deeper look into the theory of measurement, one finds that the correct procedure is to combine the uncertainties as follows:

$$\sqrt{(4s)^2 + (3s)^2} = 5s,$$

So

$$(18 \pm 4) \text{ s} + (23 \pm 3) \text{ s} = (41 \pm 5) \text{ s}.$$

In general, the uncertainty, A, connected with the addition or subtraction of N quantities with uncertainties $\pm a_i$ is

$$A = \sqrt{\sum_{i=1}^N a_i^2}$$

To find the combined uncertainty of a product or quotient, the procedure is to first express the individual uncertainties as fractional uncertainties and then combine them as above to get a combined fractional uncertainty. Note that the fractional uncertainties are dimensionless. That means they carry no unit like m or s. For example,

$$(15 \pm 1) \text{ m} / (7.0 \pm 0.4) \text{ s} = (2.1 \pm ?) \text{ m/s}.$$

The fractional uncertainty in the length measurement is:

$$S_l = \frac{1m}{15m} = 0.07$$

and in the time measurement it is:

$$S_t = \frac{0.4s}{7.0s} = 0.06.$$

Thus

$$S = \sqrt{S_l^2 + S_t^2} = \sqrt{(0.07)^2 + (0.06)^2} = 0.09$$

is the combined fractional uncertainty. Now we can calculate the uncertainty by simply multiplying S with the value 2.1 m/s:

$$0.09 \times 2.1m/s = 0.2m/s$$

So, the final result is given by:

$$(15 \pm 1) m / (7.0 \pm 0.4) s = (2.1 \pm 0.2) m/s.$$

To find the fractional uncertainty in a quantity raised to a power, the fractional uncertainty in the quantity is multiplied by the power to get the fractional uncertainty in the result. Thus, if

$$s = aT^4$$

and the fractional uncertainty in T is 0.1; there will be a fractional uncertainty in s of 0.4.

Procedure:

Do exercises as provided by the laboratory instructor.

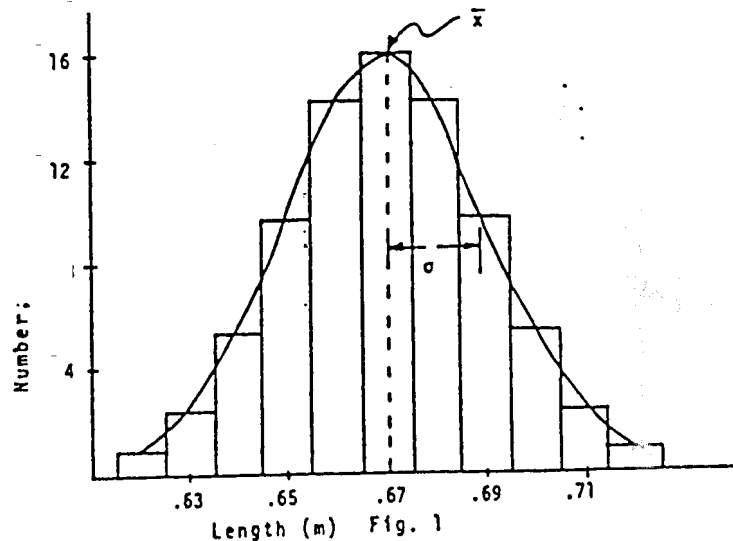
The Normal Distribution:

In a number of physics experiments, a single measurement is not adequate, so the measurement is repeated many times to reduce the uncertainty. If the cause of the differences in these individual measurements is random, i.e. varies in an uncontrollable way, then the most probable value for the result is the mean value of all the measurements. (This is not necessarily the case for other distributions where the mean, median and mode do not coincide.)

Suppose that the drop of a ping pong ball from a certain height was being measured with a stop watch. As the experiment is repeated, tiny density fluctuations in air, or tiny breezes, may affect the drop time. Also judging exactly when the ball strikes the floor will vary with each instance. These and other uncontrollable effects may make the measurement either too large or too small for a particular drop, so it doesn't seem unreasonable to take the mean value as the most probable result. (It is assumed that no "systematic errors" such as the clock running too slow or too fast are present.)

But how much uncertainty should be attached to the mean value? The theory of measurement asserts that for a large number of repeated measurements with random uncertainties, the measurements form

a “normal distribution” which is illustrated in Fig. 1 for an ideal sample of 80 measurements of a length. The value of the measurement is plotted against the number of times it occurs in the sample of 80. The smooth curve in the figure is the limiting shape of the histogram for a very large number of measurements.



Knowing the ideal distribution curve for a series of measurements, it is usual to define an uncertainty as follows. The amount of uncertainty will be such that the probability of any new measurement falling in the interval from the mean value minus the uncertainty to the mean value plus the uncertainty will be about 67 %. This definition of uncertainty is called the “standard deviation of the mean” and will be denoted by σ . It is shown in Fig. 1.

So with N measurements the mean, \bar{x} , is found by

$$\bar{x} = \frac{\sum_{i=1}^N x_i}{N}$$

and, to a good approximation,

$$\sigma = \sqrt{\frac{\sum_{i=1}^N (x_i - \bar{x})^2}{N}}$$

Procedure:

1. Each person in the lab should measure the period of the pendulum with the timer several times so that a total of about 40 measurements are made. Each student should measure the period for only one complete cycle then pass the stopwatch on to the next person until the watch has gone around the room enough times to make 40 measurements. Data should be recorded to 0.01 s, for the best results try to keep the amplitude of the pendulum swing consistent all the measurements, even by restarting as necessary.
2. Using all this data compute the mean value of the period and the standard deviation of the mean.
3. Plot a histogram similar to Fig.1 using the data obtained in (1). In order to create a histogram you will need to “bin” your data. You will divide the range of measured values into 10 intervals or “bins”. 10 is chosen in order to get reasonable number of data points (n) in each bin or interval. Firstly divide the total

