Electricity and Magnetism

Answer any four problems. Do not turn in solutions for more than four problems. Each problem has the same weight.

1. Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ/ϵ_0), as shown in the figure.

(a) Find the electric field between the spheres by the Gauss's law.

(b) Calculate the surface-charge distribution on the inner sphere.

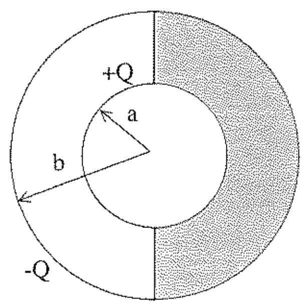

Fig. Q1

2. Consider a circular current with a radius a flowing on a conducting coil. (a) Obtain the magnetic fields on axis by Biot-Savart's law. The coil width is negligible. (b) Place a small circular coil with a radius $b(b \ll a)$ along a common axis, and separated by a distance d (Fig. Q2). Obtain the mutual inductance between two coils. The magnetic field inside the second coil is assumed to be uniform.

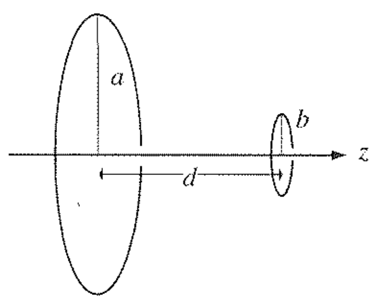

Fig. Q2

3. Consider a spherical conductor with a radius R and magnetic permeability μ placed at the origin in vacuum (μ_0). Now impose the uniform magnetic fields H_0 in the $z-$direction. (a) Calculate the magnetostatic potential distribution $\phi_m(r,\theta)$ inside and outside the sphere. Here θ is the angle from the z-axis. (b) Obtain the magnetic field H_{in} inside the sphere, and show when $\mu \gg \mu_0$, H_{in} becomes very small inside the sphere in comparison of H_0.

4. Find the potential inside and outside a spherical shell of radius R (Fig. Q4), which carries a uniform surface charge, by the following two different methods. The total charge in the sphere is Q. The inside/outside of the sphere is vacuum with ϵ_0

(a) Obtain the potential $\phi(r)$ by the Gausss law, and draw the radial potential profile, r vs. $\phi(r)$.

(b) Obtain the potential $\phi(r)$ by applying the general solution of Poissons equation, and show $\phi(r)$ is consistent with one from the Gauss's law. The general solution of Poisson's equation is

$$\phi(x) = \frac{1}{4\pi\epsilon}\int_{\infty}^{\infty}\frac{\rho_e(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x,$$

here $\mathbf{x}$ is the observation point and $\mathbf{x}'$ is the charge position. The integration is performed for all charge locations.

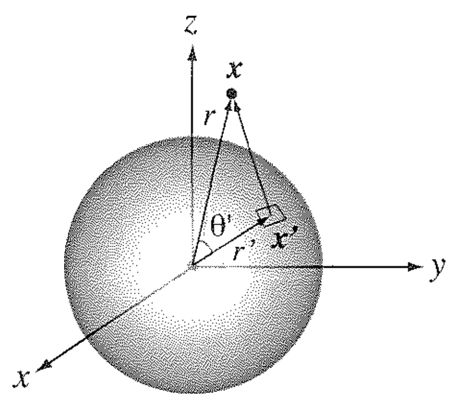

Fig. Q4

5. Consider a linear antenna of a straight wire with a length L along the z-axis. Now stimulate the antenna by imposing an oscillating current. The oscillating current is given as a function of

$$\mathbf{i}(\mathbf{r},t) = \frac{\mathbf{r}}{|\mathbf{r}|}\frac{I}{2}\sin\frac{2\pi z}{L}\delta(x)\delta(y)e^{-i\omega t}.$$

(a) Calculate the time-averaged angular distribution of radiation.
(b) Determine the time-averaged total power of radiation.

Hints of Electricity and Magnetism

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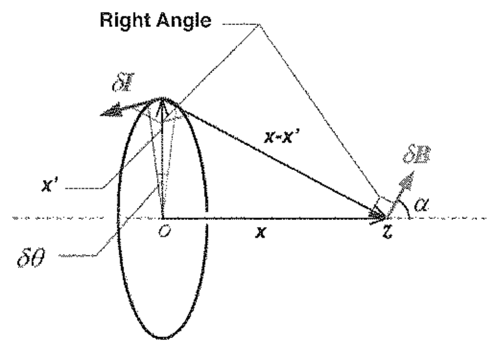

Fig. Q2 (hint)

Q3. The general solution of the Laplace equation of the magnetostatic potential ϕ_m for the spherical coordinates with azimuthal symmetry is

$$\phi_m(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta),$$

here A_l and B_l are coefficient determined by the boundary conditions, and $P_l(x)$ is the Legendre polynomials,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad \cdots.$$

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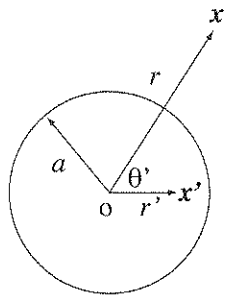
$$\phi(r) = \frac{1}{4\pi\epsilon}\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' \frac{\rho(r',\theta',\varphi')}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}.$$

An uniform surface charge density is given as $\rho(r',\theta',\varphi') = \sigma_0\delta(r' - R)$ in the this problem. Here $\delta(x)$ is the delta function.



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Q5. The time-averaged angular distribution of radiation is calculated by

$$\bar{\mathbf{S}} \cdot \mathbf{n} = \frac{\mu_0}{32\pi^2 c}\frac{|\ddot{\mathbf{p}}|^2}{r^2}\sin^2\theta,$$

here $\bar{\mathbf{S}}$ is the time-averaged pointing flux, and $\ddot{\mathbf{p}}$ is the second derivative of the dipole moment in time. The time-averaged total power of radiation through a surface S is calculated as

$$P = \int_S \bar{\mathbf{S}} \cdot \mathbf{n} dS.$$

User the following integration formula,

$$\int_{-L/2}^{L/2} \frac{z}{|z|} \sin\frac{2\pi z}{L} dz = 2L/\pi.$$

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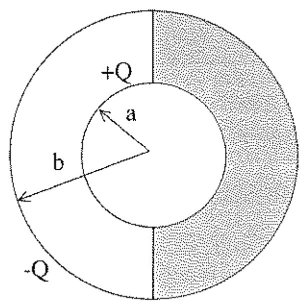

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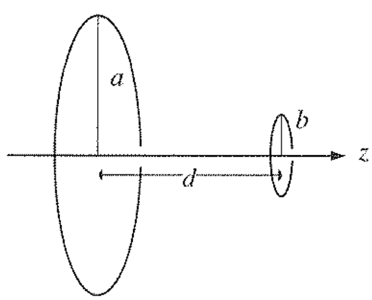

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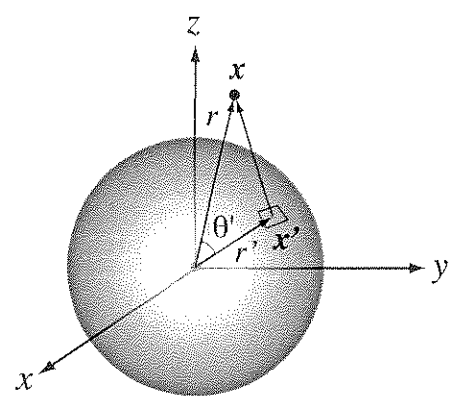

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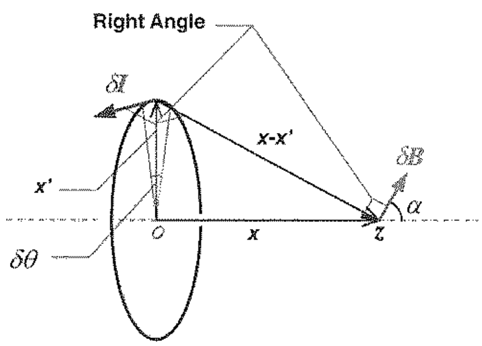

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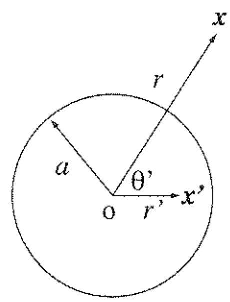

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Please solve four of the problems

Statistical Mechanics
January 2011

Useful constants: $h = 6.626 \times 10^{-34}$ J s, $k_B = 1.381 \times 10^{-23}$ J/K, $m_e = 9.110 \times 10^{-31}$ kg, $m_n = 1.675 \times 10^{-27}$ kg, $N_A = 6.022 \times 10^{23}$ mol^{-1}, 1 eV $= 1.602 \times 10^{-19}$ J.

1. *Properties of a system of boson excitations.* At low temperature, $T <$ 1 K, a system of quantized excitations called ripplons resides on the surface of a liquid (area $A = L \times L$). These excitations are bosons and have frequency $\omega = \omega_0(qa)^{\frac{3}{2}}$ and energy $\hbar\omega$, where the wavevector, q, spans the range $0 \leq q \leq a^{-1}$ ($\omega_0 = 10^{13}$ rad/sec and $a = 10^{-8}$ cm).

   (a) What is the energy of an excitation with $q = a^{-1}$? What is its equivalent temperature in degrees Kelvin?

   (b) At $T = 1$ K what is the magnitude of an excited wavevector?

   (c) How does the wavevector from (b) compare with the largest possible wavevector?

   (d) For low temperatures, $T < 1$ K, find how the mean energy of the system of excitations scales with temperature. Hint: Based on (c), state why the upper limit on the energy integral may be approximated as infinity; then write the integral dimensionlessly.

   (e) Find how the specific heat of the system of excitations scales with temperature.

2. *Calculation of a partition function.* A system of N independent, distinguishable particles is in equilibrium with a temperature reservoir at T. Each particle can be in one of 4 energy levels $-\epsilon$, 0, 0 and ϵ. The energy level 0 is 2-fold degenerate.

   (a) Calculate the partition function for this system.

   (b) Find the Helmholtz free energy for this system.

   (c) What dimensionless variable controls the behavior of this system, e.g., determines whether it is at high temperature or low temperature?

   (d) What is the entropy of this system at high temperature?

   (e) What is the entropy of this system at low temperature?

3. *Properties of a system of fermion excitations at low T.* The neutrons in a neutron star have the nominal number density (number/cm^3) of an atomic nucleus. For parts (a)-(e) assume the system is at low temperature.

   (a) What is the number density of the neutrons? (order of magnitude)

   (b) What is the fermi energy of the neutrons (order of magnitude)? Give the answer in MeV, Kelvin, and in units of $m_n c^2$, where m_n is the rest mass of a neutron.

   (c) Are the neutrons relativistic, $v/c > 0.1$?

   (d) In a neutron star there are protons at density of about 5% the density of the neutrons and there are electrons present to balance the protons (charge neutrality). Calculate the fermi energy of the electrons.

   (e) Are the electrons relativistic, $v/c > 0.1$?

   (f) The interior temperature of a neutron star is at $T \approx 10^7$ K. Is a neutron star at low temperature or high temperature? Why?

4. *Behavior of a system at co-existence.* Near $T = 1$ K and $P = 10$ MPa the phases 1 and 2 of material X co-exist. The volume per particle of phase 1 is $v_1 = a^{-3}$, $a = 3 \times 10^{-8}$ cm, and the volume per particle of phase 2 is $v_2 = 0.9\, v_1$. The entropy per particle of phase 1 is given by

$$s_1 = k_B \frac{T}{T_0}, \tag{1}$$

where $T_0 = 2050$ K. Each particle in phase 2 can reside in one of two states, a or b, with energies $\epsilon_a = 10.000$ Kelvin and $\epsilon_b = 10.025$ Kelvin.

   (a) What is the entropy per particle of the particles in phase 2?

   (b) What is the slope of the coexistence curve near $T = 1$ K? (think Gibbs-Duhem)

   (c) Estimate the pressure on the coexistence curve at $T = 10$ K.

5. *Formal thermodynamic manipulations.* A system described by extensive variables U (energy), V (volume) and N (particle number) has an entropy given by

$$S = k_B \, 3 \left(\frac{N^2 U}{\epsilon_0} \right)^{\frac{1}{3}} + k_B \, \frac{N}{\gamma} \left(\frac{V}{N v_0} \right)^{\gamma}, \tag{2}$$

where k_B (the Boltzmann constant), ϵ_0 (an energy), v_0 (a volume) are known numbers. The differential form of S is

$$dS = \frac{1}{T} dU + \frac{P}{T} dV - \frac{\mu}{T} dN, \tag{3}$$

where T is the temperature, P is the pressure and μ is the chemical potential. Take N to be fixed.

(a) Find the $1/T$ and P/T equations of state.

(b) Express S as a function of T and V, i.e., find $S(T, V)$.

(c) Write out dS for $S(T, V)$ ($dN = 0$) and find the coefficients of dT and dV.

(d) Find the specific heat at constant V,

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_V . \tag{4}$$

(e) Find the specific heat at constant P,

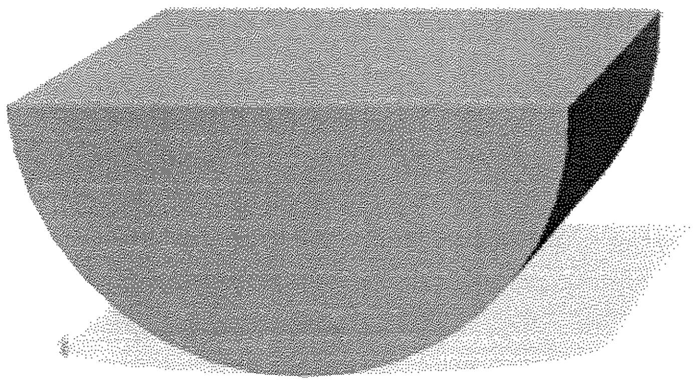
$$C_P = T \left(\frac{\partial S}{\partial T} \right)_P . \tag{5}$$

(f) Find $C_P - C_V$. What limits on γ are implied by your result?

Comprehensive exam – Classical Mechanics – 2011 Spring Semester

Please solve four of the following problems

Problem 1. The figure shows a uniform, solid half-cylinder of mass *M* and radius *R* resting on a horizontal surface. If one side of this cylinder is pushed down slightly and then released, the half-cylinder will perform small oscillations about its equilibrium position. Determine the period of these oscillations. The moment of inertia of the whole cylinder (i.e. mass *2M*) about its axis is $I = MR^2$.



Problem 2. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α are positive constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

Problem 3. A hoop of mass *m* and radius *R* rolls without slipping down an inclined plane of mass *M*, which makes an angle α with the horizontal. Find the Lagrange equations and the integrals of motion if the plane can slide without friction along a horizontal surface. The moment of inertia of the hoop with respect to its axis is $I = mR^2$.

Problem 4. Assume the Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does the Earth then have? Will the Earth escape the solar system?

Problem 5. An automobile drag racer drives a car with acceleration *a* and instantaneous velocity *V*. The tires have radius *R* and are not slipping. Find which point on the tire has the greatest acceleration relative to the ground. What is this acceleration?

Comprehensive Examination
Department of Physics University of Nevada, Reno
January 15, 2011; 9:00 -11:30 am

QUANTUM THEORY

Answer any four problems. Do not turn in solutions for more than four problems.
Each problem has the same weight.

1. Bound state of a particle in a "delta function" potential.
Consider a particle whose Hamiltonian $\hat{H}$ is:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)$$

where α is a positive constant whose dimensions are to be found and $\delta(x)$ is Dirac's delta function.

a) Integrate the eigenvalue equation of $\hat{H}$ between $-\varepsilon$ and $+\varepsilon$. Letting ε approach 0, show that the derivative of the eigenfunction $\phi(x)$ presents a discontinuity at $x = 0$ and determine it in terms of α, m and $\phi(0)$.

b) Assume that the energy E of the particle is negative (bound state). $\phi(x)$ can then be written:

$$\begin{array}{lll} x < 0 & & \phi(x) = A_1 e^{\rho x} + A_1' e^{-\rho x} \\ x > 0 & & \phi(x) = A_2 e^{\rho x} + A_2' e^{-\rho x} \end{array}$$

Express the constant ρ in terms of E and m. Using the results of the preceding question, calculate the matrix M defined by:

$$\begin{pmatrix} A_2 \\ A_2' \end{pmatrix} = M \begin{pmatrix} A_1 \\ A_1' \end{pmatrix}$$

Then, using the condition that $\phi(x)$ must be square-integrable. find the possible values of the energy. Calculate the corresponding normalized wave functions.

2. Consider the Hamiltonian H of a particle in a one-dimensional problem defined by:

$$H = \frac{1}{2m}P^2 + V(X)$$

where X and P are operators which satisfy the relation: $[X, P] = i\hbar$. The eigenvectors of H are denoted by $|n\rangle$:

$$H|n\rangle = E_n|n\rangle$$

, where n is a discrete index.

a. Show that:

$$\langle n|P|n'\rangle = \alpha\langle n|X|n'\rangle$$

where α is a coefficient which depends on the difference between E_n and $E_{n'}$. Calculate α (hint: consider the commutator $[X, H]$).

b. From this, deduce, using the closure relation, the equation:

$$\sum_{n'} (E_n - E_{n'})^2 \left|\langle n|X|n'\rangle\right|^2 = \frac{\hbar^2}{m^2}\langle n|P^2|n'\rangle.$$

3. Consider a particle whose Hamiltonian is

$$\hat{H} = \begin{pmatrix} 2 & i & 0 \\ -i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) Is $|\psi\rangle = \begin{pmatrix} i \\ 7i \\ -2 \end{pmatrix}$ an eigenstate of $\hat{H}$? Is $\hat{H}$ Hermitian?

b) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the normalized energy eigenvectors $|1\rangle, |2\rangle, |3\rangle$ of $\hat{H}$.

c) Find the matrix corresponding to the projector $\hat{P} \equiv |1\rangle\langle 1|$.

d) Calculate the commutator $\left[\hat{P}, \hat{H}\right]$ using matrix products.

4. Consider a spin 1/2 particle placed in a magnetic field B_0 with components

$$B_x = \frac{1}{\sqrt{2}}B_0, \qquad B_y = 0, \qquad B_z = \frac{1}{\sqrt{2}}B_0$$

The particle has the magnetic moment $\vec{M} = \gamma\vec{S}$, and the spin space is spanned by the vectors $|+\rangle$ and $|-\rangle$. (In this problem we are only interested in the spin properties.)

a.) In the $\{|+\rangle, |-\rangle\}$ basis, calculate the matrix representing the operator $\hat{H} = -\vec{M}\cdot\vec{B}$, the Hamiltonian of the system.

b.) Calculate the eigenvalues and the eigenvectors of $\hat{H}$.

5. HARD SPHERE SCATTERING: Scattering by a constant, spherical, repulsive potential of range R and infinite height.

(a) Determine the phase shift and the scattering amplitude for s-wave scattering. Draw a qualitative sketch of the radial wave function to illustrate the notion of a phase shift.

(b) How does the problem change if we want to determine the phase shift and the scattering amplitude for higher angular momenta? What is the wave function for $\ell \neq 0$?

(c) What is the value of the logarithmic derivative at R ?

(d) Determine the differential cross section for the case of very low projectile energy. For this limiting case we better remember the small argument forms of the spherical Bessel and Neumann functions

$$j_l(kr) \rightarrow \frac{(kr)^{l+1}}{(2l+1)!!}$$
$$n_l(kr) \rightarrow \frac{(2l-1)!!}{(kr)^l}$$

The spherical Bessel and Neumann functions, j_l and n_l , are related to the standing wave solutions $F_l(kR)$ and $G_l(kR)$ of the FREE scattering equation,

$$\left(\frac{d^2}{dr^2}+k^2\right)\Psi = \frac{\ell(\ell+1)}{r^2}\Psi.$$

(e) Determine also the total cross section for this case and compare your result to the geometric cross section which you would expect from classical mechanics. (Hint: The projectile is considered to be a point particle)

(f) Give a good argument why, for very low scattering energy, only the s-wave makes a noticeable contribution to the differential cross section.

6. Consider two distinguishable particles, each with intrinsic spin $\frac{1}{2}$, which move in a uniform external magnetic field $\mathbf{B}$. Between the two particles exists an interaction of the form $A\frac{\hbar^2}{4}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ where A is a constant and $\boldsymbol{\sigma}_1$and $\boldsymbol{\sigma}_2$ are the Pauli spin operators of the two particles. The magnetic moments of the two particles are $a\boldsymbol{\sigma}_1$ and $b\boldsymbol{\sigma}_2$, respectively. Find the *exact* energy eigenvalues of the system.

The Pauli matrices, in the standard representation, are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$